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THREE-MULTI-INDEX MITTAG-LEFFLER FUNCTIONS, SERIES AND CONVERGENCE THEOREMS ⋆

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Abstract: In this paper we consider a new class of special functions, namely, the so-called three-multi-index Mittag-Leffler functions. They are $3m$-index generalizations of the classical Mittag-Leffler functions $E_{\alpha,\beta}$ and of the Prabhakar function $E^{\gamma}_{\alpha,\beta}$. We study the basic properties of these entire functions: we find their order and type, an asymptotic estimation, represent them as Wright's generalized hypergeometric functions and Fox's H-functions. Formulae for integer and fractional order integration and differentiation are provided. We present also some interesting particular cases of the three multi-index Mittag-Leffer functions.
Series in such kind of functions are then studied in the complex plane. More precisely, their domains of convergence are found and the behaviour of such expansions on the boundary of their domains is studied. In this way, we find analogues of the Cauchy-Hadamard, Abel, Tauber and Hardy-Littlewood theorems for the power series.

Keywords: 3m-parametric multi-index Mittag-Leffer functions, order and type of entire function, asymptotic formula, Mellin-Barnes-type integral representation, Riemann-Liouville fractional integral and derivative, convergent series, summation of divergent series

1. INTRODUCTION

Recently the interest in Mittag-Leffler's (M-L) functions E_α and $E_{\alpha,\beta}$

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\;\; E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{1.1}$$

$$\alpha>0,\ \beta>0,$$

([3], Sec. 18.1) has grown up extensively due to their role in the Fractional Calculus (FC) as solutions of fractional order differential and integral equations that model important process in the real physical and social world. Therefore some their generalizations have appeared nowadays.

For example, *Prabhakar* [20] generalized them by introducing the 3-parametric function $E^{\gamma}_{\alpha,\beta}$

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\quad \alpha,\ \beta,\ \gamma\in\mathbb{C},\quad Re(\alpha)>0, \tag{1.2}$$

where $(\gamma)_k$ is the *Pochhammer symbol* ([3], Section 2.1.1) $(\gamma)_0=1,\quad (\gamma)_k=\gamma(\gamma+1)\dots(\gamma+k-1),\quad k=1,2,\dots,$ and studied some of its properties.

It is evidently that for $\gamma=1$ this function coincides with $E_{\alpha,\beta}$, and for $\gamma=\beta=1$ with E_α, i.e.:

$$E^{1}_{\alpha,\beta}(z)=E_{\alpha,\beta}(z),\quad E^{1}_{\alpha,1}(z)=E_\alpha(z).$$

Lately a class of special functions of M-L type that are multi-index analogues of $E_{\alpha,\beta}(z)$ has been introduced and studied by *Kiryakova* (see e.g. [7 - 9]). There, the indices $\alpha:=1/\rho$, $\beta:=\mu$ are replaced by two sets of multi-indices $\alpha\to(1/\rho_1,1/\rho_2,...,1/\rho_m)$, and $\beta\to(\mu_1,\mu_2,...,\mu_m)$

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\ \dots\ \Gamma\left(\mu_m+\frac{k}{\rho_m}\right)}. \tag{1.3}$$

The same functions, considered also by *Luchko* [11] and *Yakubovich and Luchko* [27] are called Mittag-Leffler functions of vector index.

In this paper we consider the new, recently introduced multi-index ($3m$-parametric) M-L function [19], generalizing on one hand the function $E^{\gamma}_{\alpha,\beta}$ and on the other hand the multi-index M-L functions (1.3). We derive some basic properties of this function, defined for complex parameters, including its order and type as entire function, its place among the other known special functions, differentiation and integration of integer and fractional order, and study the convergence of series in them in the complex plane.

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2. DEFINITIONS AND BASIC PROPERTIES OF THE 3M-PARAMETRIC MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

Definition 2.1. *Let $m > 1$ be an integer and for all $i = 1, 2, \ldots m$ the parameters $\alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{C}$, and $Re(\alpha_i) > 0$. By means of the multi-indices (α_i), (β_i), (γ_i) we introduce the so-called 3m-parametric multi-index Mittag-Leffler (3m- multi- M-L) functions, namely*

$$E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \ldots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)} \frac{z^k}{(k!)^m}. \tag{2.1}$$

Theorem 2.1. ([19]) *If each of the parameters $\gamma_1, \ldots, \gamma_m$ is neither a negative integer nor zero (i.e. $\gamma_i \notin \mathbb{Z}_0^-$), then the multi-index Mittag-Leffler function (2.1) is an entire function of order ρ and type σ with*

$$\begin{aligned} 1/\rho &= Re(\alpha_1) + \cdots + Re(\alpha_m), \\ 1/\sigma &= |(\rho\alpha_1)^{\rho\alpha_1}| \ldots |(\rho\alpha_m)^{\rho\alpha_m}|. \end{aligned} \tag{2.2}$$

Moreover, for each positive ε the asymptotic estimate

$$|E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z)| < \exp((\sigma + \varepsilon)|z|^{\rho}), \quad |z| \geq r_0 > 0 \tag{2.3}$$

holds with ρ and σ like in (2.2) for $|z| \geq r_0(\varepsilon)$, $r_0(\varepsilon)$ sufficiently large.

Theorem 2.2. ([19]) *If at least one of the parameters $\gamma_1, \ldots, \gamma_m$ is a non-positive integer, then the multi-index Mittag-Leffler function (2.1) reduces to a finite sum.*

Most of the Special Functions (SF) of mathematical physics are special cases of the generalized hypergeometric functions ${}_pF_q$, and thus, of the more general Meijer $G-$functions ([3], Vol.1, Ch.5). However, the M-L functions serve as an example of SF that could not be included in the scheme of the Meijer $G-$functions, being a more general Fox's $H-$function. Namely,

$$\begin{aligned} E_{\alpha,\beta}(z) &= H^{1,\,1}_{1,\,2}\left[-z \left| \begin{matrix} (0,\ 1) \\ (0,\ 1), (1-\beta,\ \alpha) \end{matrix} \right.\right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds \end{aligned} \tag{2.4}$$

and only for rational $\alpha = p/q$, (2.4) reduces to a $G-$function.

Therefore, it is important to emphasize the place of the introduced function (2.1) among the known SF, especially Wright's generalized hypergeometric function and Fox's H-function (see e.g. [5, 6 (Appendix), 21, 26]).

Definition 2.2. *By a Fox H-function we mean a generalized hypergeometric function defined by means of the Mellin-Barnes-type contour integral*

$$H^{m,n}_{p,q}\left[\sigma \left| \begin{matrix} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right.\right] = \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{\prod_{i=1}^{m} \Gamma(b_i - sB_i) \prod_{j=1}^{n} \Gamma(1 - a_j + sA_j)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i + sB_i) \prod_{j=n+1}^{p} \Gamma(a_j - sA_j)} \sigma^s ds, \tag{2.5}$$

where $\mathcal{L}'$ is a suitable contour in $\mathbb{C}$, $m,\ n,\ p,\ q$ are integers $0 \leq m \leq q$, $0 \leq n \leq p$, the parameters $a_j, b_i \in \mathbb{C}$, $A_j, B_i > 0$, $j = 1, \ldots, p$, $i = 1, \ldots, q$ and $A_j(b_i + l) \neq B_i(a_j - l' - 1)$, $l, l' = 0, 1, 2, \ldots$.

For $A_1 = \cdots = A_p = 1$, $B_1 = \cdots = B_q = 1$ (2.5) turns into the more popular Meijer's $G-$function (see [3], Vol. 1, Ch. 5, 23, 14). The $G-$ and $H-$ functions encompass almost all the elementary and special functions and this makes the knowledge on them very useful. The Wright generalized hypergeometric functions ${}_p\Psi_q$ with irrational $A_j, B_i > 0$, give examples of $H-$functions, not reducible to $G-$functions (e.g. representation (2.4)) and

$$\begin{aligned} {}_p\Psi_q &\left[\left. \begin{matrix} (a_1,\ A_1) \ldots (a_p,\ A_p) \\ (b_1,\ B_1) \ldots (b_q,\ B_q) \end{matrix} \right| \sigma \right] \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + kA_1) \ldots \Gamma(a_p + kA_p)}{\Gamma(b_1 + kB_1) \ldots \Gamma(b_q + kB_q)} \frac{\sigma^k}{k!} \\ &= H^{1,\,p}_{p,\,q+1}\left[-\sigma \left| \begin{matrix} (1-a_1,\ A_1), \ldots, (1-a_p,\ A_p) \\ (0,1), (1-b_1,\ B_1), \ldots, (1-b_q,\ B_q) \end{matrix} \right.\right]. \end{aligned} \tag{2.6}$$

Theorem 2.3. *Let $\alpha_i > 0$, $\beta_i, \gamma_i \in \mathbb{C}$, $Re(\gamma_i) > 0$ for $i = 1, \ldots, m$. Then the multi-index Mittag-Leffler functions (2.1) are Wright's generalized hypergeometric functions as well as Fox's H-function of the form*

$$\begin{aligned} &E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z) \\ &= A\ {}_m\Psi_{2m-1}\left[\left. \begin{matrix} (\gamma_1,\ 1), \ldots, (\gamma_m,\ 1) \\ (\beta_1,\ \alpha_1), \ldots, (\beta_m,\ \alpha_m), (1,1), \ldots, (1,1) \end{matrix} \right| z \right] \\ &= A\ H^{1,\,m}_{m,\,2m}\left[-z \left| \begin{matrix} (1-\gamma_1,\ 1), \ldots, (1-\gamma_m,\ 1) \\ [(0,1), (1-\beta_i,\ \alpha_i)]_1^m \end{matrix} \right.\right], \end{aligned} \tag{2.7}$$

with

$$A = \left[\prod_{i=1}^{m} \Gamma(\gamma_i)\right]^{-1}.$$

They have the following Mellin-Barnes-type contour integral representation, extending the integral formula (2.4):

$$\begin{aligned} E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(z) &= \frac{A}{2\pi i} \int_{\mathcal{L}'} \frac{\Gamma(-s) \prod_{i=1}^{m} \Gamma(\gamma_i + s)}{[\Gamma(1+s)]^{m-1} \prod_{i=1}^{m} \Gamma(\beta_i + s\alpha_i)} (-z)^s ds \\ &= \frac{A}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s) \prod_{i=1}^{m} \Gamma(\gamma_i - s)}{[\Gamma(1-s)]^{m-1} \prod_{i=1}^{m} \Gamma(\beta_i - s\alpha_i)} (-z)^{-s} ds, \quad z \neq 0, \end{aligned} \tag{2.8}$$

where $\mathcal{L}$ is an arbitrary contour in $\mathbb{C}$ running from $-i\infty$ to $+i\infty$ in a way that the poles $s = 0, -1, \ldots$ of $\Gamma(s)$ lie to the left of $\mathcal{L}$ and the poles of $\Gamma(\gamma_i - s)$ $(i = 1, \ldots, m)$ to the right of it.

For details of the proof, see [19].

The representation (2.7) of the multi-index M-L functions as Fox's H- functions allow to describe their asymptotic behaviour as $z \to 0$; as well as $z \to \infty$.

3. SPECIAL CASES OF THE 3M-PARAMETRIC MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

Let us mention some interesting special cases of the introduced multi- M-L functions.

1. If $m = 1$, formula (2.1) gives the *3-parametric M-L function* $E^{\gamma}_{\alpha,\beta}$ (*Prabhakar*), i. e.

$$E^{\gamma}_{\alpha,\,\beta}(z) = E^{(\gamma),\,1}_{(\alpha),\,(\beta)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}$$
$$= \frac{1}{\Gamma(\gamma)}\, {}_1\Psi_1 \left[\begin{matrix} (\gamma,\ 1) \\ (\beta,\ \alpha) \end{matrix} \middle| z \right].$$

In addition, if $\gamma = 1$,

$$E_{\alpha,\,\beta}(z) = E^{(1),\,1}_{(\alpha),\,(\beta)}(z), \quad E_{\alpha}(z) = E^{(1),\,1}_{(\alpha),\,(1)}(z),$$
$$E_1(z) = \exp(z) = E^{(1),\,1}_{(1),\,(1)}(z).$$

2. If $\gamma_1 = \cdots = \gamma_m = 1$, the definition (2.1) gives the *multi-index M-L functions (of* $2m$*-indices), introduced and studied in details by Kiryakova*, more precisely, putting in (2.1) $1/\rho_i$ instead of α_i, $\alpha_i = 1/\rho_i > 0$, we obtain:

$$E_{\left(\frac{1}{\rho_i}\right),(\beta_i)}(z) = E^{(1),\,m}_{\left(\frac{1}{\rho_i}\right),\,(\beta_i)}(z)$$
$$= E^{(1,\ldots,1),\,m}_{\left(\frac{1}{\rho_i}\right),\,(\beta_i)}(z) = {}_1\Psi_m \left[\begin{matrix} (1,\ 1) \\ (\beta_i,\ \alpha_i)_1^m \end{matrix} \middle| z \right], \tag{3.1}$$

and moreover the formulae (2.2) and (2.3) reduce to these obtained by *Kiryakova* [7].

For $m \geq 2$, denote $\beta_i = \delta_i + 1, i = 1, \ldots, m$, and let additionally $\rho_1 = \cdots = \rho_m = 1$ and one of the β_i to be 1, e.g.: $\beta_m = 1$, i.e. $\delta_m = 0$. Then the multi-M-L function (3.1) becomes a *hyper-Bessel function*, in the sense of *Delerue*, see details in [6] (App., (D.30) and Ch.3):

$$J^{(m-1)}_{\delta_1,\ldots,\delta_{m-1}}(z) = B\ E^{(1,\ldots,1),\,m}_{(1,\ldots,1),\,(\delta_1+1,\ldots,\delta_{m-1}+1,\,1)}(-(z/m)^m)$$
$$= BC\ {}_0F_{m-1}\left(\delta_1+1,\ldots,\delta_{m-1}+1;\ -\left(\frac{z}{m}\right)^m\right)$$

with

$$B = \left(\frac{z}{m}\right)^{\sum\limits_{i=1}^{m-1}\delta_i}, \quad C = \left[\prod_{i=1}^{m-1}(\delta_i+1)\right]^{-1}.$$

The last functions are closely related to the hyper-Bessel operators, introduced by Dimovski [1], see also [6].

For $m = 2$ the functions (3.1) is *Dzrbashjan's M-L* type functions from [2], denoted by $E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}$, i.e.

$$E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}(z) = E^{(1,1),\,2}_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}(z).$$

3. A special case (for $m \geq 2$) is the *generalized Lommel-Wright function with 4 indices* ($\mu > 0, q \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$), *introduced by de Oteiza, Kalla and Conde*, see [6] (Appendix):

$$J^{\mu,\,q}_{\nu,\,\lambda}(z) = (z/2)^{\nu+2\lambda} E^{(1,1,\ldots,1),\,q+1}_{(\mu,1,\ldots,1),\,(\nu+\lambda+1,\,\lambda+1,\ldots,\,\lambda+1)}\left(-(z/2)^2\right)$$
$$= (z/2)^{\nu+2\lambda}\ {}_1\Psi_{q+1} \left[\begin{matrix} (1,\ 1) \\ (\lambda+1,\ 1)_1^q, (\lambda+\nu+1,\ \mu) \end{matrix} \middle| -(z/2)^2 \right]. \tag{3.2}$$

This is an interesting example of a multi-index M-L function (3.1) with arbitrary $m = q + 1$.

Some other interesting cases are given below.

Obviously, for $q = 1$, the special function (3.2) turns into the generalization of the Bessel function $J_\nu(z)$, *introduced by Pathak* (for details see [6], Appendix):

$$J^{\mu}_{\nu,\,\lambda}(z) = (z/2)^{\nu+2\lambda} E^{(1,1),\,2}_{(\mu,1),\,(\nu+\lambda+1,\,\lambda+1)}\left(-(z/2)^2\right).$$

For particular choices of the other parameters λ and μ we obtain results for more special cases.

Let $\lambda = 0$, then the special function (3.2) gives the generalization of the Bessel-Clifford function $C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z})$, *introduced by Wright* and called *Bessel-Wright or Bessel-Maitland function* (see [6, 8]):

$$J^{\mu}_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+\mu k+1)} = E^{(1,1),\,2}_{(\mu,1),(\nu+1,1)}(-z)$$
$$= {}_0\Psi_1 \left[\begin{matrix} -- \\ (\nu+1,\ \mu) \end{matrix} \middle| -z \right]. \tag{3.3}$$

Initially, Wright defined (3.3) only for $\mu > 0$, then extended its definition to $\mu > -1$.

Additionally, if $\mu = 1$, then (3.2) get to the *classical Bessel function* ([3], Vol. 2):

$$J_\nu(z) = (z/2)^{\nu}\, E^{(1,1),\,2}_{(1,1),\,(\nu+1,\,1)}\left(-(z/2)^2\right).$$

4. FRACTIONAL RIEMANN-LIUOVILLE (R-L) INTEGRAL AND DERIVATIVE

The notion Fractional Calculus (FC) or Fractional Analysis is used for the extension of the Calculus (Analysis), when the order of integration and differentiation can be an arbitrary number (fractional, irrational, complex), that is, not obligatory integer. For its theory and applications, see the *FC encyclopedia* [24]. The most popular definition for integration of order $\lambda \in \mathbb{C}$ ($Re(\lambda) > 0$), is the *Riemann-Liouville (R-L) fractional integral*

$$R^{\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t)dt$$
$$= \frac{z^{\lambda}}{\Gamma(\lambda)} \int_0^1 (1-\tau)^{\lambda-1} f(z\tau)d\tau\,. \tag{4.1}$$

Then, the *R-L fractional derivative* of order $\lambda \in \mathbb{C}$ ($Re(\lambda) > 0$), is defined as a composition of a derivative of integer order and an integral of fractional order of the form (4.1), i.e.:

$$D^{\lambda} f(z) := D^n\, R^{n-\lambda} f(z), \tag{4.2}$$

where $n := [Re(\lambda)] + 1 > Re(\lambda)$, $[Re(\lambda)]$ = integer part of $Re(\lambda)$.

In this section we consider the R-L fractional integrals and derivatives (4.1) and (4.2) of order $\lambda \in \mathbb{C}$ ($Re(\lambda) > 0$), of the multi- M-L function (2.1). To this purpose, we need first the derivatives of integer order. The differentiation of the multi- M-L function (2.1) is given by the following elementary assertion, [19].

Lemma 4.1. *Let* $\alpha_i,\ \beta_i,\ \gamma_i,\ \omega \in \mathbb{C}$, *and* $Re(\alpha_i)$, $Re(\beta_{i_0})$ *are positive numbers,* $i = 1, \ldots, m$, $1 \leq i_0 \leq m$, $i_0 \in \mathbb{N}$, *then for any* $n \in \mathbb{N}$ *the following identity holds:*

$$D^n[z^{\beta_{i_0}-1} E^{(\gamma_i),\,m}_{(\alpha_i),\,(\beta_i)}(\omega z^{\alpha_{i_0}})]$$

$$= \left(\frac{d}{dz}\right)^n [z^{\beta_{i_0}-1} E^{(\gamma_i),\, m}_{(\alpha_i),\,(\beta_i)}(\omega z^{\alpha_{i_0}})]$$

$$= z^{\beta_{i_0}-n-1} E^{(\gamma_i),\, m}_{(\alpha_i),\,(\widetilde{\beta}_i)}(\omega z^{\alpha_{i_0}}) \tag{4.3}$$

with $\widetilde{\beta}_i = \beta_i$, *if* $i \neq i_0$, *and* $\widetilde{\beta}_{i_0} = \beta_{i_0} - n$.

Note 4.1. In particular, as a result of (4.3), we obtain the following relations:

$$D^n[z^{\beta-1} E^{\gamma}_{\alpha,\,\beta}(\omega z^{\alpha})] = z^{\beta-n-1} E^{\gamma}_{\alpha,\,\beta-n}(\omega z^{\alpha}),$$

$$D^n[z^{\beta_{i_0}-1} E^{m}_{(\alpha_i),\,(\beta_i)}(\omega z^{\alpha_{i_0}})] = z^{\beta_{i_0}-n-1} E^{m}_{(\alpha_i),\,(\widetilde{\beta}_i)}(\omega z^{\alpha_{i_0}}),$$

with $\widetilde{\beta}_i = \beta_i$, if $i \neq i_0$, and $\widetilde{\beta}_{i_0} = \beta_{i_0} - n$. These results follow if $m = 1$, respectively, $\gamma_1 = \cdots = \gamma_m = 1$.

Theorem 4.1. *Let* $\alpha_i,\ \beta_i,\ \gamma_i,\ \omega,\ \lambda \in \mathbb{C}$, *and* $Re(\alpha_i),\ Re(\beta_{i_0}),\ Re(\lambda) > 0,\ i = 1,\ldots,m,\ 1 \le i_0 \le m$, $i_0 \in \mathbb{N}$, *then the following identity holds:*

$$R^{\lambda}[z^{\beta_{i_0}-1} E^{(\gamma_i),\, m}_{(\alpha_i),\,(\beta_i)}(\omega z^{\alpha_{i_0}})] = z^{\beta_{i_0}+\lambda-1} E^{(\gamma_i),\, m}_{(\alpha_i),\,(\widetilde{\beta}_i)}(\omega z^{\alpha_{i_0}}) \tag{4.4}$$

with $\widetilde{\beta}_i = \beta_i$, *if* $i \neq i_0$, *and* $\widetilde{\beta}_{i_0} = \beta_{i_0} + \lambda$.

Theorem 4.2. *Let* $\alpha_i,\ \beta_i,\ \gamma_i,\ \omega,\ \lambda \in \mathbb{C}$, *and* $Re(\alpha_i),\ Re(\beta_{i_0}),\ Re(\lambda) > 0,\ i = 1,\ldots,m,\ 1 \le i_0 \le m$, $i_0 \in \mathbb{N}$, *then the following identity holds:*

$$D^{\lambda}[z^{\beta_{i_0}-1} E^{(\gamma_i),\, m}_{(\alpha_i),\,(\beta_i)}(\omega z^{\alpha_{i_0}})] = z^{\beta_{i_0}-\lambda-1} E^{(\gamma_i),\, m}_{(\alpha_i),\,(\widetilde{\beta}_i)}(\omega z^{\alpha_{i_0}}) \tag{4.5}$$

with $\widetilde{\beta}_i = \beta_i$, *if* $i \neq i_0$, *and* $\widetilde{\beta}_{i_0} = \beta_{i_0} - \lambda$.

For the proofs, see [19].

5. SERIES IN PRABHAKAR AND $2M$-MULTI-INDEX MITTAG-LEFFLER FUNCTIONS

Explicit solutions of some kinds of fractional order (or multi-order) differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions (as for example, in Kiryakova and Al-Saqabi [10] and Sandev, Tomovski and Dubbeldam [25]). This fact provokes the studying of series in Mittag-Leffler type functions like those considered in this work. First, we are dealing with series in functions of the type (1.3) and then with (1.2).

To this end, let us fix $1 \le i_0 \le m$ and for the sake of brevity, set

$$\alpha_i = 1/\rho_i,\ \ (\alpha_i > 0,\ i = 1, 2, ..., m)$$

$$(\alpha_{i_0}(n)) = (\alpha_1, \ldots, \alpha_{i_0-1}, n, \alpha_{i_0+1}, \ldots, \alpha_m) = (\alpha_i)\,|_{\alpha_{i_0}=n}$$

$$(\mu_{i_0}(n)) = (\mu_1, \ldots, \mu_{i_0-1}, n, \mu_{i_0+1}, \ldots, \mu_m) = (\mu_i)\,|_{\mu_{i_0}=n}$$

$$\prod_{i=1}^{m}{}' \,\Gamma(\alpha_i k + \mu_i) = \prod_{i=1,\ i \neq i_0}^{m} \Gamma(\alpha_i k + \mu_i).$$

Consider now the multi- M-L functions with indices of the kind (α_i), $(\mu_{i_0}(n))$ and $(\alpha_{i_0}(n))$, (μ_i).

Remark 5.1. Depending on α_i and μ_i $(i = 1, 2, \ldots, m)$, some coefficients in (1.2) and (1.3) may be equal zero, that is possible only when some of the numbers μ_i are non-positive real numbers, but no more than finite number of the coefficients might be zero. That is, there exist numbers $p, q, r \in \mathbb{N}_0$ such that the identities (1.3) can be written as

$$E_{(\alpha_i),\,(\mu_{i_0}(n))}(z) = \sum_{k=p}^{\infty} \frac{z^k}{\Gamma(\alpha_{i_0} k + n) \prod_{i=1}^{m}{}' \,\Gamma(\alpha_i k + \mu_i)},$$

for $n \in \mathbb{N}_0$ $(\mu_{i_0} = n)$, and respectively as

$$E_{(\alpha_{i_0}(n)),\,(\mu_i)}(z) = \sum_{k=q}^{\infty} \frac{z^k}{\Gamma(nk + \mu_{i_0}) \prod_{i=1}^{m}{}' \,\Gamma(\alpha_i k + \mu_i)},$$

for $n \in \mathbb{N}$ $(\alpha_{i_0} = n)$.

We introduce the following auxiliary functions, connected with the three parametric Mittag-Leffler functions (1.2), namely:

$$\gamma = 0:\ \widetilde{E}^{0}_{\alpha,\,0}(z) = 0, \quad \widetilde{E}^{0}_{\alpha,\,n}(z) = \Gamma(n)\, z^n\, E^{0}_{\alpha,\,n}(z),\ n \in \mathbb{N},$$

$$\gamma \neq 0:\ \widetilde{E}^{\gamma}_{\alpha,\,n}(z) = \frac{\Gamma(\alpha r + n)}{(\gamma)_r}\, z^{n-r}\, E^{\gamma}_{\alpha,\,n}(z),\ \ n \in \mathbb{N}_0,$$

and with the $2m$-multi-index functions (3.1), namely:

$$\widetilde{E}_{(\alpha_i),\,(\mu_{i_0}(n))}(z) = z^{n-p}\, E_{(\alpha_i),\,(\mu_{i_0}(n))}(z) \prod_{i=1}^{m}{}' \,\Gamma(\alpha_i p + \mu_i),$$

for $n \in \mathbb{N}_0$, respectively

$$\widetilde{E}_{(\alpha_{i_0}(n)),\,(\mu_i)}(z) = z^{n-q}\, E_{(\alpha_{i_0}(n)),\,(\mu_i)}(z) \prod_{i=1}^{m}{}' \,\Gamma(\alpha_i q + \mu_i),$$

for $n \in \mathbb{N}$.

and consider series in these functions, respectively of the form:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,\,n}(z), \tag{5.1}$$

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_{(\alpha_i),\,(\mu_{i_0}(n))}(z) \tag{5.2}$$

$$\sum_{n=1}^{\infty} a_n \widetilde{E}_{(\alpha_{i_0}(n)),\,(\mu_i)}(z) \tag{5.3}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

For the above considered series (5.1) - (5.3) we give theorems, corresponding to the classical Cauchy-Hadamard, Abel, Tauber and Littlewood theorems for the power series.

The same type convergence theorems have been also obtained for series in other special functions, for example, for series in Laguerre and Hermite polynomials, by [22 - 23], and resp. by the author – for series in other representatives of (2.1), namely Bessel functions and their Wright's 2-, 3-, and 4-indices generalizations, in [12 - 15], and in Mittag-Leffler functions, in [16, 17]. The results for the series (5.2), (5.3) have been published in [18].

6. CAUCHY-HADAMARD AND ABEL TYPE THEOREMS

In the beginning, we give a theorem of Cauchy-Hadamard type for the series (5.1) – (5.3).

Theorem 6.1. (of Cauchy-Hadamard type). *The domain of convergence of each of the series (5.1) – (5.3) with complex coefficients* a_n *is the disk* $|z| < R$ *with a radius of convergence* $R = 1/\Lambda$, *where*

$$\Lambda = \limsup_{n\to\infty} (\,|a_n|\,)^{1/n}. \tag{6.1}$$

for the series (5.1),

$$\Lambda = \limsup_{n\to\infty} \left(\frac{|a_n|}{\Gamma(\alpha_{i_0} p + n)} \right)^{1/n} \tag{6.2}$$

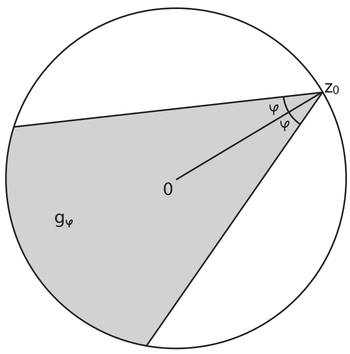
for the series (5.2), and

$$\Lambda = \limsup_{n\to\infty} \left(\frac{|a_n|}{|\Gamma(nq + \mu_{i_0})|} \right)^{1/n} \tag{6.3}$$

for the series (5.3).
The cases $\Lambda = 0$ *and* $\Lambda = \infty$ *can be included in the general case, provided* $1/\Lambda$ *means* ∞, *respectively 0.*

The ideas of the proofs for the series (5.2) and (5.3) can be seen in [18]. The proof for the series (5.1) follows the same lines.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0.



The following theorem is valid.

Theorem 6.2. (of Abel type). *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers,* Λ *be the real number defined by (6.1), respectively (6.2), (6.3),* $0 < \Lambda < \infty$. *Let* $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$. *If* $f(z;\alpha,\gamma)$, $g(z;\alpha_{i_0})$, $h(z;\mu_{i_0})$ *are the sums respectively of the series (5.1), (5.2), (5.3) on the domain* K, *and these series converge at the point* z_0 *of the boundary of* K, *then:*

$$\lim_{z\to z_0} f(z;\alpha,\gamma) = \sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,n}(z_0), \tag{6.4}$$

$$\lim_{z\to z_0} g(z;\alpha_{i_0}) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{(\alpha_i),(\mu_{i_0}(n))}(z_0), \tag{6.5}$$

$$\lim_{z\to z_0} h(z;\mu_{i_0}) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{(\alpha_{i_0}(n)),(\mu_i)}(z_0), \tag{6.6}$$

provided $|z| < R$ *and* $z \in g_\varphi$.

The details of the proofs for the equalities (6.5) and (6.6) can be seen in [18]. The proof for the equality (6.4) goes in the similar way.

7. (J, Z_0)-SUMMATIONS AND TAUBERIAN TYPE THEOREMS

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0,1,2,... \tag{7.1}$$

To define its Abel summability ([4], p. 20, 1.3 (2)), we consider also the power series

$$\sum_{n=0}^{\infty} a_n z^n. \tag{7.2}$$

Definition 7.1. The series (7.1) is called A - *summable*, if the power series (7.2) converges in the unit disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$ and moreover there exists the limit

$$\lim_{z\to 1-0} \sum_{n=0}^{\infty} a_n z^n = S.$$

The complex number S is called A-sum of the series (7.1) and the usual notation of that is

$$\sum_{n=0}^{\infty} a_n = S \qquad (A).$$

Let $z_0 \in \mathbb{C}$, $\quad z_0 \neq 0$, $\quad |z_0| = R$, $\quad 0 < R < \infty \quad$ and
$(J; z_0) := \{j_n : \; j_n - \text{entire function}, \; j_n(z_0) = 1\}_{n\in\mathbb{N}_0}$

To define the (J, z_0)- summation of the numerical series (7.1), we consider the series of the kind (7.3) instead of (7.2):

$$\sum_{n=0}^{\infty} a_n j_n(z), \quad j_n \in (J; z_0). \tag{7.3}$$

Definition 7.2. The numerical series (7.1) is said to be (J, z_0) - *summable* if the series (7.3) converges in the disk $|z| < R$ and, moreover, there exists the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n j_n(z),$$

provided z remains on the segment $[0, z_0)$.

Let $z_0 \in \mathbb{C}$ with $|z_0| = R$, $0 < R < \infty$, $\widetilde{E}^{\gamma}_{\alpha,n}(z_0) \neq 0$, $\widetilde{E}_{(\alpha_i),(\mu_{i_0}(n))}(z_0) \neq 0$ and $\widetilde{E}_{(\alpha_{i_0}(n)),(\mu_i)}(z_0) \neq 0$. For the sake of brevity, denote

$$j_n(z) := E^{*}_{\alpha,n,\gamma}(z; z_0) = \frac{\widetilde{E}^{\gamma}_{\alpha,n}(z)}{\widetilde{E}^{\gamma}_{\alpha,n}(z_0)}, \tag{7.4}$$

respectively

$$j_n(z) := E^{\star}_{(\alpha_i),(\mu_{i_0}(n))}(z; z_0) = \frac{\widetilde{E}_{(\alpha_i),(\mu_{i_0}(n))}(z)}{\widetilde{E}_{(\alpha_i),(\mu_{i_0}(n))}(z_0)}, \quad n \in \mathbb{N}_0, \tag{7.5}$$

or

$$j_n(z) := E^{\star}_{(\alpha_{i_0}(n)),(\mu_i)}(z; z_0) = \frac{\widetilde{E}_{(\alpha_{i_0}(n)),(\mu_i)}(z)}{\widetilde{E}_{(\alpha_{i_0}(n)),(\mu_i)}(z_0)}, \quad n \in \mathbb{N}, \tag{7.6}$$

and consider the corresponding series of the kind (7.3), i.e.:

$$\sum_{n=0}^{\infty} a_n E^{*}_{\alpha,\,n,\,\gamma}(z;z_0), \tag{7.7}$$

respectively

$$\sum_{n=0}^{\infty} a_n E^{\star}_{(\alpha_i),\,(\mu_{i_0}(n))}\,(z;z_0), \tag{7.8}$$

or

$$\sum_{n=1}^{\infty} a_n E^{\star}_{(\alpha_{i_0}(n)),\,(\mu_i)}\,(z;z_0). \tag{7.9}$$

Remark 7.1. The (J, z_0) - summation is regular for every one of the above considered cases, and this property is just a particular case of Theorem 6.2.

Let us just mention that the Tauber theorem is a statement that relates the Abel summability and the standard convergency of a numerical series by means of some assumptions imposed on the general term of the series under consideration. A classical result in this direction is given in [4] (Theorem 85).

Furthermore we extend the validity of such type of assertion to series in means of just considered families (7.4) – (7.6) of multi- M-L functions. Tauber type theorems are given also for summations by means of Laguerre polynomials [22], and Bessel type functions by the author [12, 13].

Theorem 7.1. (of Tauber type). *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with $\lim\limits_{n\to\infty} na_n = 0$ and the numerical series (7.1) is (J, z_0) -summable (in a sense of the functions (7.4), respectively (7.5) or (7.6)), then the series (7.1) is summable.*

At first sight it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeds to weaken it and obtain the strengthened version of the Tauber theorem ([4], Theorem 90).

A Littlewood generalization of the $o(n)$ version of Tauber type theorem (Theorem 7.1) is given below. Similar theorems are proved for example in [14] and [16].

Theorem 7.2 (of Littlewood type). *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with $a_n = O(1/n)$ and the numerical series (7.1) is (J, z_0) -summable (in a sense of the functions (7.4), respectively (7.5) or (7.6)), then it is summable.*

The proofs of the last two theorems in a sense of the functions (7.5), respectively (7.6), are given in [18]. In a sense of the functions (7.4), the theorems can be proved analogously using a specific asymptotic estimate, but that will be published elsewhere.

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